PMM U.S.S.R.,Vol.54,No.1,pp.60-63,1990
Printed in Great Britain

0021-8928/90 $10.00+0.00

ON SEEKING BUCKLING MODES OF A CIRCULAR PLATE*

A.A. FONAREV

A construction of the buckling modes of a circular plate is examined by using the solutions of an infinite system of non-linear algebraic equations that appears on substituting their non-trivial solution representable, by assumption, in the form of series, into the non-linear Karman equations. It is shown that an approximation can be found to the solution of the system by using a projection method and a projection-iteration process in the Banach space of sequences whose series from the elements converge absolutely. Results of computations are presented.

1. The axisymmetric deformation of a thin circular elastic plate of constant thickness that is in equilibrium under a uniform compressive load applied along an edge is described by the non-linear Karman equations /1/ that reduce to the following system of equations

$$GQ(r) + \lambda^2 (1 - P(r)) Q(r) = 0,\ GP(r) = -{}^1/{}_2 Q^2(r),\ 0 < r < 1 \tag{1.1}$$

$$G = r^{-3} d(r^3 d/dr)/dr$$

where r is the dimensionless radius λ^2 is a dimensionless load parameter, Q is the dimensionless derivative of the transverse displacement with respect to the radius, and $(P - 1)$ is the dimensionless radial stress.

The assumption on symmetry and smoothness reduces to the conditions

$$Q'(0) = 0,\ P'(0) = 0 \tag{1.2}$$

If the edge $r = 1$ of the plate is rigidly clamped, then the additional boundary conditions

$$Q(1) = 0,\ P(1) = 0 \tag{1.3}$$

should be satisfied.

For any λ the boundary-value problem (1.1)-(1.3) has the trivial solution $Q(r) \equiv 0$, $P(r) \equiv 0$ (the non-buckling mode). Other (non-trivial) real solutions are called buckling modes.

As a result of linearization of problem (1.1)-(1.3) near the non-buckling mode, a linear second-order boundary-value problem is obtained

$$G\bar{Q} + \lambda^2 \bar{Q} = 0,\ 0 < r < 1;\ \bar{Q}'(0) = \bar{Q}(1) = 0,\ \bar{P} \equiv 0$$

which, for $\lambda = \lambda_n$ has the non-trivial solutions

$$\bar{Q}_n = r^{-1} J_1(\lambda_n r),\ J_1(\lambda_n) = 0\ (n = 1, 2, \ldots)$$

utilized later to construct the plate buckling modes, where λ_n is the n-th zero of the Bessel function J_1.

It is known that buckling modes exist for $\lambda > \lambda_1$ (see /2/ and the bibliography in /2/, say).

We assume that the non-trivial solution $Q(r)$ and $P(r)$ of problem (1.1)-(1.3) is represented by the series

$$Q(r) = \varepsilon \Sigma_a,\quad P(r) = \varepsilon^2 \Sigma_b;\ \Sigma_a = \sum_{n=1}^{\infty} a_n \bar{Q}_n,\quad a_1 = 1,\quad \Sigma_b = \sum_{n=1}^{\infty} b_n \bar{Q}_n \tag{1.4}$$

where ε belongs to the neighbourhood of the zero of the real line R, $\varepsilon \neq 0$. Then substitution of series (1.4) into the first equation in (1.1) yields

*Prikl.Matem.Mekhan.,54,1,75-79,1990

$$\sum_{n=2}^{\infty} a_n (\lambda^2 - \lambda_n^2) \bar{Q}_n + (\lambda^2 - \lambda_1^2) \bar{Q}_1 - \lambda^2 \varepsilon^2 \Sigma_a \Sigma_b = 0$$

After multiplying this equality by $r^3 \bar{Q}_m(r)$ and integrating between 0 and 1 with respect to r, the following expressions are obtained because of the orthogonality of $\bar{Q}_1, \bar{Q}_2, \ldots$:

$$a_n = \frac{\lambda^2 \varepsilon^2 I_n}{(\lambda^2 - \lambda_n^2) \| \bar{Q}_n \|^2} \quad (n = 2, 3, \ldots), \quad \lambda^2 = \lambda_1^2 \frac{\| \bar{Q}_1 \|^2}{\| \bar{Q}_1 \|^2 - \varepsilon^2 I_1} \tag{1.5}$$

$$I_n = \int_0^1 r^3 \Sigma_a \Sigma_b \bar{Q}_n \, dr, \quad \| \bar{Q}_n \|^2 = \int_0^1 r J_1^2 (\lambda_n r) \, dr = \frac{1}{2} J_0^2 (\lambda_n) \quad (n = 1, 2, \ldots)$$

Similarly, the expressions

$$b_n = \frac{1}{2 \lambda_n^2 \| \bar{Q}_n \|^2} \int_0^1 r^3 \Sigma_a^2 \bar{Q}_n \, dr \quad (n = 1, 2, \ldots) \tag{1.6}$$

are obtained on substituting series (1.4) into the second equation of (1.1).

Expressions (1.5) and (1.6) yield an infinite system of non-linear algebraic equations in $a_2, a_3, \ldots$ and $b_1, b_2, \ldots$ which will be investigated later. It will be shown here that the system has a solution in a certain right semicircle of the point λ_1.

If $a_2, a_3, \ldots$ and $b_1, b_2, \ldots$ in series (1.4) are solutions of system (1.5) and (1.6), then series (1.4) yield a non-trivial solution (or the buckling mode) of problem (1.1)-(1.3) with load parameter λ^2 obtained on substituting the solution of system (1.5) and (1.6) into the right-hand side of the second formula in (1.5). The solution of problem (1.1)-(1.3) being obtained here formally satisfies (1.1) because the convergence of series (1.4) being obtained, and the series for derivatives of terms of these series are not investigated.

If the boundary-value problem (1.1)-(1.3) is studied in the neighbourhood of the point $\lambda = \lambda_m$ $(m \geqslant 2)$, and not $\lambda = \lambda_1$, then this case is considered analogously, and it is sufficient to take $a_m = 1$ instead of $a_1 = 1$ in (1.4).

2. Let us investigate the infinite system of non-linear algebraic Eqs.(1.5) and (1.6).

Let l_1 be a Banach space of sequences $x = (x_1, x_2, \ldots)$ or real numbers for which the series $|x_1| + |x_2| + \ldots$ with the norm $\|x\| = |x_1| + |x_2| + \ldots$ converges, and let $D \equiv \{x = (x_1, x_2, \ldots) \in l_1 \colon x_1 = 1, \ |x_2| + |x_3| + \ldots \leqslant C\}$, where C is an arbitrary fixed positive number.

We consider the mapping B from D in the set of sequences that sets the sequence $Bx = ((Bx)_1, (Bx)_2, \ldots)$ obtained on substituting elements of the sequence x into the right-hand side of (1.6) in place of $a_1, a_2, \ldots$ for $n = 1, 2, \ldots$, into correspondence with the element $x = (x_1, x_2, \ldots) \in D$. For each $n \geqslant 1$ we have

$$|(Bx)_n| \leqslant \left(\frac{1 + C}{\lambda_n J_0 (\lambda_n)} \right)^2 \int_0^1 | J_1 (\lambda_n r) | \, dr \leqslant \frac{3^{3/4} (1 + C)^2}{2 \lambda_n^2 | J_0 (\lambda_n) |^{3/2}}$$

because we have $|J_1(t)| \leqslant 1$ for all $t \geqslant 0$, which, when the Hölder inequality is utilized, yields

$$\int_0^1 | J_1 (\lambda_n r) | \, dr \leqslant \int_0^1 r^{-1/4} (r^{1/4} | J_1 (\lambda_n r) |^{1/2}) \, dr \leqslant$$

$$\left(\int_0^1 r^{-1/3} \, dr \right)^{3/4} \left(\int_0^1 r J_1^2 (\lambda_n r) \, dr \right)^{1/4} = \frac{3^{3/4}}{2} | J_0 (\lambda_n) |^{1/2}, \quad \forall n$$

Therefore, by virtue of the convergence of the series

$$\sum_{n=1}^{\infty} \lambda_n^{-2} | J_0 (\lambda_n) |^{-3/2} \tag{2.1}$$

(this series converges because $J_0(\lambda_n) = (2/(\pi\lambda_n))^{1/2} (\cos (\lambda_n - \pi/4) + O(\lambda_n^{-1}))$ and $\lambda_n = (n + 1/4)\pi + O(n^{-1})$ for large values of n, see /3/) $Bx \in l_1$ and a constant $C_1 > 0$, independent of $x \in D$ exists such that $\| Bx \| \leqslant C_1$.

Furthermore, for all $x, y \in D$ we have

$$|(Bx-By)_n| \leqslant 2\frac{1+C}{\lambda_n^2 J_0^2(\lambda_n)} \int_0^1 \Big(\sum_{i=2}^{\infty} |(x_i-y_i) J_1(\lambda_i r)|\Big) |J_1(\lambda_n r)| \, dr \leqslant$$
$$3^{3/4} \frac{1+C}{\lambda_n^2 |J_0(\lambda_n)|^{3/2}} \|x-y\|$$

for $n \geqslant 1$. This means that a constant $C_2 > 0$ exists such that $\|Bx - By\| \leqslant C_2\|x-y\|$, $\forall x, y \in D$.

Let $l_1 \times R$ be the Banach space of the pair $(x, t) \in l_1 \times R$, $x \in l_1$ and $t \in R$, with the norm $\|(x, t)\| = \|x\| + |t|$.

We define a real-valued function T that sets a number $T(x, t)$ obtained on substituting elements of the sequences x and Bx in place of $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$, respectively, and t in place of ε on the right-hand side of the second formula of (1.5) in correspondence with the element $(x, t) \in D \times R$ $(x \in D)$ in $D \times R \subset l_1 \times R$. Positive numbers ε_1, C_3 and C_4 exist such that $|T(x, \varepsilon)| \leqslant C_3$ and $|T(x, \varepsilon) - T(y, \varepsilon)| \leqslant C_4\varepsilon^2\|x-y\|$ for all $\varepsilon \in [-\varepsilon_1, \varepsilon_1]$ and $x, y \in D$.

We also define the mapping A from $D \times R$ into the set of sequences that sets the sequence $A(x, t) = ((A(x, t))_1, (A(x, t))_2, \ldots)$ with $(A(x, t))_1 = 1$ and $(A(x, t))_n$ $(n \geqslant 2)$ obtained by substituting elements of the sequences x and Bx in place of $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$, respectively, and the numbers $T(x, t)$ and t in place of λ^2 and ε, respectively, in the right-hand side of (1.5) for $n = 2, 3, \ldots$, in correspondence with the element $(x, t) \in D \times R$. Constants $\varepsilon_2 \in (0, \varepsilon_1]$, $C_5 > 0$, $C_6 > 0$, exist such that $A(x, \varepsilon) \in l_1$, $\|A(x, \varepsilon)\| \leqslant 1 + C_5\varepsilon^2$ and $\|A(x, \varepsilon) - A(y, \varepsilon)\| \leqslant C_6\varepsilon^2\|x-y\|$ for all $\varepsilon \in [-\varepsilon_2, \varepsilon_2]$ and $x, y \in D$. Therefore, constants $\varepsilon_0 \in (0, \varepsilon_2]$ and $q \in (0, 1)$ exist such that $A(x, \varepsilon) \in D$ and $\|A(x, \varepsilon) - A(y, \varepsilon)\| \leqslant q\|x - y\|$ for all $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ and $x, y \in D$. And the mapping $A(x, \varepsilon)$ is here continuous in ε in $D \times [-\varepsilon_0, \varepsilon_0]$.

We note that explicit expressions can be obtained for the constants C_i $(i = 1, \ldots, 6)$ in terms of C and the sum of the series (2.1).

Because of the fixed-point principle /4/ a mapping x_0: $[-\varepsilon_0, \varepsilon_0] \to D$ exists such that $A(x_0(\varepsilon), \varepsilon) = x_0(\varepsilon)$ for all $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$, where such a mapping is unique and continuous. And for each fixed $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ the element $x_0(\varepsilon)$ is the limit of the sequence $v^{i+1} = A(v^i, \varepsilon)$ $(i = 0, 1, \ldots)$ with arbitrary $v^0 \in D$. Furthermore, if elements of the sequences $x_0(\varepsilon)$ and $Bx_0(\varepsilon)$ are taken as $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ in the series (1.4), then these series yield the solution of the boundary-value problem (1.1)-(1.3) with the load parameter $\lambda^2 = T(x_0(\varepsilon), \varepsilon)$ $(\varepsilon \in [-\varepsilon_0, \varepsilon_0])$.

System (1.5) and (1.6) cannot be successfully solved exactly; consequently, approximate methods for solving are considered later.

We fix any $i \geqslant 2$ and consider the subspace E_i of the space l_1 consisting of the sequences $x = (x_1, x_2 \ldots) \in l_1$ with $x_n = 0$ for all $n > i$. We define the linear projection operator F_i: $l_1 \to E_i$, that sets the element $F_i x = y \equiv (y_1, y_2, \ldots)$ with $y_n = x_n$ for $n = 1, \ldots, i$ and $y_n = 0$ for $n > i$ in correspondence with the element $x = (x_1, x_2, \ldots) \in l_1$. We have $\|F_i x\| \leqslant \|x\|$, $\forall x \in l_1$.

We introduce the set $D_i = D \cap E_i$ and the mapping $B_i : D \to E_i$, $B_i x = F_i B(F_i x)$ for $x \in D$.

We define the function T_i: $D \times [-\varepsilon_0, \varepsilon_0] \to R$, that sets the number $T_i(x, t)$ obtained on substituting the elements of the sequences $F_i x$ and $B_i x$ in place of $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$, respectively, and t in place of ε in the left-hand side of the second formula in (1.5) in correspondence with the element $(x, t) \in D \times [-\varepsilon_0, \varepsilon_0]$. We also define the mapping A_i: $D \times [-\varepsilon_0, \varepsilon_0] \to E_i$ that sets an element $A_i(x, t) = z \equiv (z_1, z_2, \ldots) \in E_i$ with $z_1 = 1$ and z_n $(n = 2, \ldots, i)$ obtained as a result of substituting elements of the sequences $F_i x$ and $B_i x$ in place of $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$, and the numbers $T_i(x, t)$ and t in place of λ^2 and ε, respectively, in the right-hand side of the first formula in (1.5) for $n = 2, \ldots, i$, in correspondence with the element $(x, t) \in D \times [-\varepsilon_0, \varepsilon_0]$. We have $A_i(x, \varepsilon) \in D_i$ and $\|A_i(x, \varepsilon) - A_i(y, \varepsilon)\| \leqslant q\|x-y\|$ for all $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ and $x, y \in D$, where the number q is the same as above. Therefore, from the theorem in /5/ we obtain the following theorem on the convergence of the sequences of the projection method (i.e., the convergence of $x^i(\varepsilon)$) and the projection-iteration process (i.e., the convergence of $y^i(\varepsilon)$) that combines the projection method and the iteration process in one, to the solution $x_0(\varepsilon)$ of the equation $A(x, \varepsilon) = x$ $(x \in D)$.

Theorem. For each $i \geqslant 2$ a mapping x^i: $[-\varepsilon_0, \varepsilon_0] \to D_i$, exists such that $A_i(x^i(\varepsilon), \varepsilon) = x^i(\varepsilon)$ for all $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$, where such a mapping is unique and continuous; the sequence of mappings $x^i(\varepsilon)$ $(i = 2, 3, \ldots)$ converges uniformly to $x_0(\varepsilon)$ in $[-\varepsilon_0, \varepsilon_0]$; for any mapping y^1: $[-\varepsilon_0, \varepsilon_0] \to D_2$ a sequence of mappings $y^{i+1}(\varepsilon) = A_{i+1}(y^i(\varepsilon), \varepsilon)$ $(i = 1, 2, \ldots)$ converges uniformly to $x_0(\varepsilon)$ in $[-\varepsilon_0, \varepsilon_0]$.

Remark. For fixed $i \geqslant 2$ the mapping $x^i(\varepsilon)$ from the theorem can be found by using the iteration process

$$z^{j+1}(\varepsilon) = A_i(z^j(\varepsilon), \varepsilon) \quad (j = 0, 1, \ldots)$$

with the arbitrary initial mapping $z^0 : [-\varepsilon_0, \varepsilon_0] \to D_i$, where the following estimate of the rate of convergence holds:

$$\| z^j(\varepsilon) - x^i(\varepsilon) \| \leqslant 2(1 + C) q^j/(1 - q) \quad (j = 0, 1, \ldots)$$

for all $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$.

The Theorem and the Remark enable us to seek an approximation to the solution of system (1.5) and (1.6) by which approximations to the buckling mode of problem (1.1)-(1.3) can be constructed.

In conformity with the above discussion, computations were performed using the projection method nad the projection-iteration process on ES-1033 and ES-1061 computers with double precision. The results showed that ε_0 is not a small number.

We will present some results obtained by the projection method. In this case the computations were performed for fixed ε in conformity with the Remark. The computation ceased for

$$\max_{n=2,\ldots,i} |(z^{j+1}(\varepsilon) - z^j(\varepsilon))_n| \leqslant \delta$$

where δ is a given accuracy (see $z^j(\varepsilon)$ in the Remark), here an element with $(z^0(\varepsilon))_n = 0$ for $n \geqslant 2$ was taken as $z^0(\varepsilon) = ((z^0(\varepsilon))_1, (z^0(\varepsilon))_2, \ldots) \in D_i$. If m iterations were performed as a result then $z^m(\varepsilon) = (a_1^m, a_2^m, \ldots)$ and $B_i z^{m-1}(\varepsilon) = (b_1^{m-1}, b_2^{m-1}, \ldots)$ were used to construct approximations to the buckling modes of problem (1.1)-(1.3): approximations to the buckling modes were examined in the form of the sums

$$Q_i = \varepsilon \sum_{n=1}^{i} a_n^m \bar{Q}_n, \quad P_i = \varepsilon^2 \sum_{n=1}^{i} b_n^{m-1} \bar{Q}_n \tag{2.2}$$

and the following quantities were calculated:

$$\gamma_1 = \max_{k=1,\ldots,49} | GQ_i(r) + \lambda^2 (1 - P_i(r)) Q_i(r) | |_{r=r_k}$$
$$\gamma_2 = \max_{k=1,\ldots,49} | GP_i(r) + {}^1/_2 Q_i^2(r) | |_{r=r_k}$$
$$(\lambda^2 = T_i(z^{m-1}(\varepsilon), \varepsilon), \ r_k = k/50)$$

which it is natural to designate as errors because they are obtained on substituting the approximations (2.2) to the buckling modes into (1.1).

Results of computations for $\delta = 10^{-15}$ are presented in the table.

ε	i	γ_1	γ_2	m	λ
0.5	55	$1.19 \cdot 10^{-5}$	$2.33 \cdot 10^{-5}$	8	3.8493
0,5	60	$8.5 \cdot 10^{-6}$	$2.1 \cdot 10^{-5}$	8	3.8493
1	60	$7.06 \cdot 10^{-5}$	$8.55 \cdot 10^{-5}$	11	3.9029
2	5	0.133	$9.95 \cdot 10^{-2}$	20	4.1300
2	20	$1.73 \cdot 10^{-2}$	$1.04 \cdot 10^{-2}$	20	4.1301
2	30	$7.1 \cdot 10^{-3}$	$4.26 \cdot 10^{-3}$	20	4.1301
2	60	$6.53 \cdot 10^{-4}$	$3.71 \cdot 10^{-4}$	20	4.1301
2,3	60	$1.06 \cdot 10^{-3}$	$5.1 \cdot 10^{-4}$	24	4.2343

REFERENCES

1. KARMAN T., Festigkeitsprobleme im Maschinenbau. Encyklopädie der Math. Wissenschaften. 4, Leipzig, 1910.
2. VOLKOVYSKY G.H., Proof of the existence of buckling modes of circular plates by using the Schauder fixed-point theorem, Bifurcation Theory and Non-linear Eigenvalue Problems, Mir, Moscow, 1974.
3. WATSON G.N., Theory of Bessel Functions, Pt. 1, Izd. Inostr. Lit., Moscow, 1949.
4. KANTOROVICH L.V. and AKILOV G.P., Functional Analysis. Nauka, Moscow, 1977.
5. FONAREV A.A., On the approximation scheme. Abstrs. of Papers Presented to the Amer. Math. Soc., 87th Annual Meeting, 2, 3, 1981.

Translated by M.D.F.